

PHYS 143 – Mechanics and Thermal Physics

Motion

*How do we describe where an object is, how it moves,
and how its motion changes?*

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By the end of this week, you should be able to:

1. Distinguish position, displacement, velocity and acceleration, and represent them with diagrams, graphs and equations.
2. Calculate velocity and acceleration by differentiation, and recover them by integration with initial conditions.
3. Select a kinematic model, and state when it is valid.
4. Solve free-fall, projectile, circular and relative-motion problems.

Central question

How do we describe where an object is, how it moves, and how its motion changes?

The thread of the week

Differentiate to go from position to velocity to acceleration; integrate, with initial conditions, to go back. Choose the simplest model that is valid, and say when it applies.

Three contacts: 1 **DESCRIBE** motion, 2 **MODEL** changing motion, 3 **APPLY** it to complex motion.

Reading: Young and Freedman, Chapter 2, §2.1–2.6 and Chapter 3, §3.1–3.5. **Labels:** **CORE** you should be able to do this; **PRACTICE** use it to develop the skill; **REFERENCE** keep it for later use. More: [hhdonglo.github.io](https://github.com/hhdonglo).

Not marked. Answer individually, then check the answers.

1. Give the signs of the x - and y -components of a 20 m/s vector at 30° above the $-x$ axis.
2. Differentiate $x = 3t^2 - 4t + 2$ and evaluate the result at $t = 2 \text{ s}$.
3. Integrate $a = 6t$, given $v = 0$ at $t = 0$.
4. Solve $4.9t^2 - 12t - 3 = 0$. Which root can be a time?
5. What do the slope and the area under a straight-line graph segment mean?
6. Convert 72 km/h to m/s .

These are the Week 1 and calculus skills this week depends on. A calculus refresher is in the supplementary notes.

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150° from $+x$: $x < 0$, $y > 0$ (-17.3 , $+10.0$).

$dx/dt = 6t - 4$, so 8 m/s.

$v = 3t^2$.

$t = \frac{12 \pm \sqrt{144 + 58.8}}{9.8} = 2.68 \text{ s or } -0.23 \text{ s; only the positive root.}$

Slope = rise/run; area = base $\times$ height (triangle: half).

$72/3.6 = 20 \text{ m/s.}$

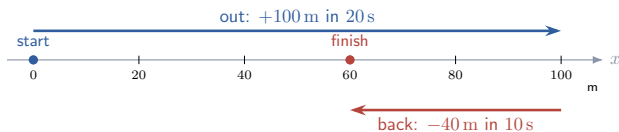
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Contact 1: Describe

How do we represent motion?

Opening problem

A runner goes 100 m east in 20 s, turns, and runs 40 m back west in 10 s. Where is the runner? Over the whole journey, how has the position changed, and how rapidly and in what direction?



Predict first

Before any calculation: is the average velocity positive or negative? Is it larger or smaller than the average speed?

But what if... “how far did the runner go?” has two honest answers, 60 m from the start or 140 m of road? We need words that tell them apart: the **vocabulary of motion**.

Quantity	What it is	What it measures	SI unit	Representation
Position	location relative to a chosen origin	where the object is	m	a point on an axis; the vector $\vec{r}$
Displacement	final position minus initial position (a vector)	how far and which way the position has changed	m	arrow from start to finish
Distance	total length of the path travelled (a scalar)	how much ground was covered	m	length along the path

Interpret: position answers *where is it?*; displacement answers *how has the position changed?*; distance answers *how much ground was covered?*

But what if... the journey also asks *how rapidly* the position changes? Words are needed for rates of change: **velocity, speed and acceleration.**

Quantity	What it is	What it measures	SI unit	Representation
Velocity	rate of change of position (a vector)	how fast, and in which direction, the position changes	m/s	tangent to the path; slope of $x-t$
Speed	the magnitude of the velocity (a scalar)	how fast, with no direction	m/s	length of the velocity arrow
Acceleration	rate of change of velocity (a vector)	how quickly the speed or the direction changes	m/s ²	arrow; slope of $v-t$

Interpret: velocity answers *how fast, and which way?*; acceleration answers *how is the velocity changing?* Each is defined precisely, with its equation, in the slides that follow.

But what if... the same word names a vector in one place and a number in another? We must fix the **notation** before using these words in equations.

Notation used in every lecture

Vector: $\vec{v}$ (bold, with an arrow). **Magnitude:** $v = |\vec{v}| \geq 0$, a number with no direction. For velocity, v is the **speed**.

Scalar component: v_x , a *signed number*: the part of $\vec{v}$ along the x axis you chose.

Component vector: $v_x \hat{i}$. The number v_x is not a vector; $v_x \hat{i}$ is.

The runner, going west

With $+x$ east:

$v_x = -4.0 \text{ m/s}$ (a signed number)

$\vec{v} = -4.0 \hat{i} \text{ m/s}$ (a vector)

$v = |\vec{v}| = 4.0 \text{ m/s}$ (the speed)

Careful: choose the axis first

v_x can be negative; the speed v cannot. Choose an origin and positive direction *before* assigning signs; reversing the axis flips every sign. A change is final minus initial.

But what if... the sign of v_x depends on which way we call positive? Then position and displacement need an **origin and an axis** chosen first.

- **Position** $x(t)$: where the object is, relative to a chosen *origin* along a chosen *axis*.

- **Displacement:**

$$\Delta x = x_2 - x_1 \quad (\text{final} - \text{initial})$$

It can be positive, negative or zero. It does not depend on the path.

- **Distance travelled:** the total path length, a scalar greater than or equal to zero.

Physical meaning

Position depends on the origin and axis you choose. Displacement is independent of the origin, provided the same coordinate system is used for both positions. Its sign depends on which direction you call positive.

Link to Week 1

In 1D, Δx is the x -component of $\Delta \vec{r} = \vec{r}_2 - \vec{r}_1$, vector subtraction from last week.

But what if... the runner turns back? $\Delta x = +60$ m but distance 140 m. How rapidly did x change in 30 s? $\rightarrow$ **average velocity**.

Definitions

$$v_{\text{av},x} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\text{average speed} = \frac{\text{distance travelled}}{\Delta t}$$

The signed $v_{\text{av},x}$ can be negative; the average speed cannot.

Back to the runner

Find the average velocity and the average speed for the whole 30 s.

Definitions

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The signed $v_{\text{av},x}$ can be negative; the average speed cannot.

Back to the runner

Find the average velocity and the average speed for the whole 30 s.

$$\Delta x = +100 - 40 = +60 \text{ m},$$

$$v_{\text{av},x} = +2.0 \text{ m/s}.$$

Distance = 140 m, so average speed
= 4.7 m/s.

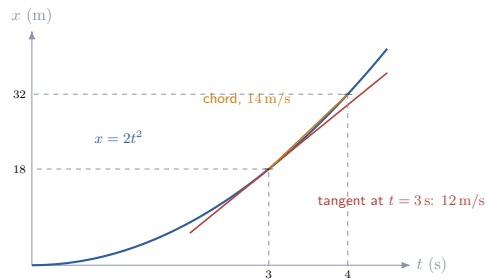
Evaluate: positive, and smaller than the average speed. The prediction holds.

But what if... the runner never actually moved at +2.0 m/s? The average hides what happened (first +5.0, then -4.0 m/s). A dragster is slower at the start of a run and faster at the end. What is the velocity at *one instant*?

A particle moves with $x = 2t^2$ (SI units).
Shrink the interval after $t = 3$ s:

Δt (s)	$v_{av,x}$ (m/s)
1.0	14.0
0.5	13.0
0.1	12.2
0.01	12.02

The chord slope approaches the tangent slope:
12 m/s.



But what if... the table only *approaches* 12 m/s and we had to guess the limit? Make the limit exact for any t : the **derivative**.

Rule: instantaneous velocity

$$v_x = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad \text{the slope of the tangent to the } x-t \text{ graph}$$

Reading the graph:

- positive slope $\rightarrow v_x > 0$ (moving in $+x$)
- negative slope $\rightarrow v_x < 0$
- zero slope $\rightarrow$ momentarily at rest
- steeper $\rightarrow$ greater speed $|v_x|$

Two different graphs

An $x-t$ graph is *not* the path of the object. The height of the graph is the position, not the speed.

But what if... we want to be sure we can use these two velocities without any algebra? Test the ideas first: **a concept check.**

Decide, and sketch an $x-t$ graph where it helps

1. Can the average speed over a journey be smaller than $|v_{av,x}|$?
2. Can the velocity be zero while the object is not at the origin?
3. Can the velocity be zero at one instant and the object still be moving a moment later?
4. Can the speed be constant while the velocity changes?

Decide, and sketch an $x-t$ graph where it helps

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Answers

1. **No.** Distance $\geq |\Delta x|$, so average speed $\geq |v_{\text{av},x}|$.
2. **Yes.** Velocity says how position is changing, not where it is.
3. **Yes.** A ball at the top of its flight has $v = 0$ only for an instant.
4. **Yes.** Constant speed on a curve: the direction of $\vec{v}$ changes (Contact 3).

But what if... the motion is not along one line, such as a ball thrown across a field? Position then needs x , y (and z): a **vector**.

In two or three dimensions the position becomes a vector:

$$\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}$$

Rule: displacement and velocity as vectors

$$\Delta \vec{r} = \vec{r}_2 - \vec{r}_1 \quad \vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}$$

Physical meaning

The velocity vector is **tangent to the path** and points in the instantaneous direction of motion. $\vec{v}_{av}$ has the direction of $\Delta \vec{r}$.

Week 1 tools

Components, signs, magnitude
 $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ and quadrants carry over.

But what if... $\vec{r}(t)$ is given by formulas for $x(t)$, $y(t)$, $z(t)$? Differentiate each component.

Rule: differentiate component by component

position $\vec{r}(t)$ $\xrightarrow{d/dt}$ **velocity** $\vec{v}(t)$ $\xrightarrow{d/dt}$ **acceleration** $\vec{a}(t)$

$\frac{d}{dt}t^n = nt^{n-1}$, $\frac{d}{dt}\cos\omega t = -\omega\sin\omega t$, $\frac{d}{dt}\sin\omega t = \omega\cos\omega t$. **Substitute** the time *after* differentiating.

When to use it

The position (or velocity) is given as formulas in t and you need the velocity (or acceleration). The reverse step, integration, comes in Contact 2.

Watch out and check

$\sin$ and $\cos$ take radians; ωt is dimensionless. Every derivative divides by seconds, so the units must change; the direction of $\vec{v}$ must match the path.

But what if... we try it on a path that curves through space? A circle in the xy -plane combined with a steady rise in z .

Question

A particle has position (SI units) $\vec{r} = 2.0 \cos(\pi t) \hat{i} + 2.0 \sin(\pi t) \hat{j} + 1.0 t^2 \hat{k}$. Find its velocity at $t = 0.5$ s.

Predict first

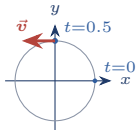
At $t = 0$ the particle is at $(2.0, 0, 0)$. In which direction is it moving at $t = 0.5$ s, and is z increasing?

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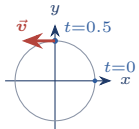
Equation: $\vec{v} = d\vec{r}/dt$. **Plan:** differentiate each term, then put $t = 0.5$ s ($\sin \frac{\pi}{2} = 1$, $\cos \frac{\pi}{2} = 0$).

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Predict first

At $t = 0$ the particle is at $(2.0, 0, 0)$. In which direction is it moving at $t = 0.5$ s, and is z increasing?



Equation: $\vec{v} = d\vec{r}/dt$. **Plan:** differentiate each term, then put $t = 0.5$ s ($\sin \frac{\pi}{2} = 1$, $\cos \frac{\pi}{2} = 0$).

Solve: $\vec{v} = -2.0\pi \sin(\pi t) \hat{i} + 2.0\pi \cos(\pi t) \hat{j} + 2.0t \hat{k}$, so at $t = 0.5$ s
 $\vec{v} = (-6.28 \hat{i} + 0 \hat{j} + 1.0 \hat{k})$ m/s, $v = \sqrt{6.28^2 + 1.0^2} = 6.36$ m/s.

Evaluate: at $t = 0.5$ s the particle is at $(0, 2.0, 0.25)$ m, the top of a circle (sketch, from $+z$); moving in $-x$ is counter-clockwise, and z increases, as predicted. Units m/s. ✓ The path is a helix.

But what if... we are given only a *graph* of $x(t)$? Then slopes must be read off the picture.

Question

Of the four points P, Q, R and S: (a) at which is v_x **most positive**? (b) at which is the **speed** greatest?
Decide first: which point has the steepest positive slope, which the steepest slope of either sign?

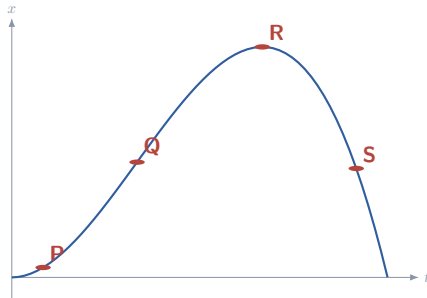
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This curve is $x = 2t^2 - t^3/3$, so $v_x = 4t - t^2$:

	t (s)	slope	v_x (m/s)
P	0.5	up	+1.75
Q	2	steepest up	+4.0
R	4	flat	0
S	5.5	steepest down	-8.25

(a) Q, (b) S: the speed is $|v_x|$, so a steep *negative* slope counts.



Common mistakes and the correct habit

Mistake

Reading an $x-t$ graph as the path of the object.

Confusing distance with displacement, speed with velocity.

Writing v when a signed component v_x is meant.

Assigning signs before choosing the axes.

Using the average velocity as the instantaneous velocity.

Subtracting initial from final the wrong way round.

Instead

The height is the position; the slope is the velocity.

Displacement and velocity are signed; distance and speed are not.

$v = |\vec{v}| \geq 0$ is the speed; v_x may be negative.

Choose the origin and the positive direction first, and mark them.

Instantaneous velocity is the slope of the tangent, not of the chord.

A change is always final minus initial.

Contact 2: Model

How do we describe changing motion mathematically?

From describing motion to modelling how it changes

The runner *turns round*: v_x goes from $+5.0$ to -4.0 m/s. **How is the velocity itself changing, and can we use that to predict the motion?**

This contact: **acceleration**; **integration**; the **constant-acceleration equations** and free fall.

Poll

Can an object have a **negative** velocity and still be **speeding up**? A car moves toward $-x$ and the driver presses the accelerator. Is v_x negative? Is it speeding up? Is a_x positive or negative?

From describing motion to modelling how it changes

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Yes. Negative velocity only says the motion is toward $-x$. Whether the object speeds up depends on how v_x and a_x are related; we make this precise in a few slides.

But what if... we try to say this without a definition of a_x itself? First we need the rate of change of velocity: **average acceleration**.

Definition

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} \quad \text{unit: m/s}^2$$

In 1D: $a_{av,x} = (v_{2x} - v_{1x})/\Delta t$.

$\Delta \vec{v} = \vec{v}_2 - \vec{v}_1 = \vec{v}_2 + (-\vec{v}_1)$: the same vector subtraction as in Week 1.

Physical meaning

Acceleration describes how the **velocity** changes: its magnitude, its direction, or both. It is not only about speed.

Back to the runner

The turn takes 2.0 s and v_x goes from +5.0 to -4.0 m/s:

$$a_{av,x} = \frac{-4.0 - 5.0}{2.0} = -4.5 \text{ m/s}^2.$$

But what if... the runner's $a_x = -4.5 \text{ m/s}^2$ is an *average* over 2.0 s? It cannot say what happens at each instant of the turn. → **instantaneous acceleration**.

Rule: instantaneous acceleration

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2 \vec{r}}{dt^2} \quad a_x = \frac{dv_x}{dt} = \frac{d^2 x}{dt^2}$$

- On a v_x - t graph, a_x is the *slope*.
- On an x - t graph, a_x is the *curvature*:
concave up $\rightarrow a_x > 0$; concave down
 $\rightarrow a_x < 0$.

The chain:

$$x(t) \xrightarrow{d/dt} v_x(t) \xrightarrow{d/dt} a_x(t)$$

But what if... the runner has $a_x < 0$ for the whole turn, yet is first *slowing down* (still moving east) and then *speeding up* (moving west)? The sign of a_x alone cannot tell us which. $\rightarrow$ **the sign rule.**

Rule: when does speed increase along a line?

Same sign of v_x and $a_x \Rightarrow$ speeding up.

Opposite signs $\Rightarrow$ slowing down.

v_x	a_x	motion
+	+	speeding up (moving toward $+x$)
+	-	slowing down
-	-	speeding up (moving toward $-x$)
-	+	slowing down

Common error

A negative acceleration does not by itself mean slowing down, and a negative velocity does not mean slowing down. Compare the two signs.

But what if... we want to see the same rule on a picture, with no numbers at all? Equal time steps make a **motion diagram**.

Question

The dots show the positions of an object at **equal time intervals**, moving from dot 1 to dot 5 with constant acceleration. At dot 3, what are the signs of v_x and a_x ? Predict first: which way is it moving, and are the spacings growing or shrinking?



equal time intervals Δt between dots; the object moves from dot 1 to dot 5

But what if... the object moves on a curve, so $\vec{v}$ changes direction as well as size? Then $\vec{a}$ has a part *along* the path and a part *across* it.

Answer: $v_x < 0$ and $a_x > 0$

The object moves toward $-x$, so $v_x < 0$ at dot 3. The spacing **shrinks**: it is slowing down, so v_x and a_x have **opposite** signs and $a_x > 0$. Check: v_x goes from a larger negative value to a smaller negative one, so $\Delta v_x > 0$.



equal time intervals Δt between dots; the object moves from dot 1 to dot 5

But what if... the object moves on a curve, so $\vec{v}$ changes direction as well as size? Then $\vec{a}$ has a part *along* the path and a part *across* it.

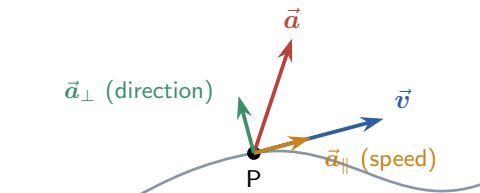
Rule: velocity and the path

$\vec{v}$ is tangent to the path.

$\vec{a}_{\parallel}$ (parallel to $\vec{v}$) changes the **speed**.

$\vec{a}_{\perp}$ (perpendicular to $\vec{v}$) changes the **direction**.

A skier who speeds up on a straight slope has only $a_{\parallel}$. A car taking a bend at constant speed has only $a_{\perp}$ (Contact 3).



But what if... we are handed a v_x-t graph with straight segments instead of a formula? Slopes give a_x , but how do we recover the displacement? $\rightarrow$ **slope and area**.

Rule: what slope and area give

Graph	slope gives	area gives
$x-t$	velocity v_x	(not used)
v_x-t	acceleration a_x	displacement Δx
a_x-t	rate of change of a_x	change in velocity Δv_x

Procedure

Translate both ways: graph $\rightarrow$ description of the motion, and description $\rightarrow$ sketched graph. Read a slope as the vertical change divided by the horizontal change, with units. Areas above the axis add; areas below subtract from the displacement but still add to the distance.

Watch out and check

The graph is not the path: an upward graph does not mean the object moves upward. Where $v_x = 0$ the object may reverse, so split the distance calculation there. Do the signs and units of the answer match the graph?

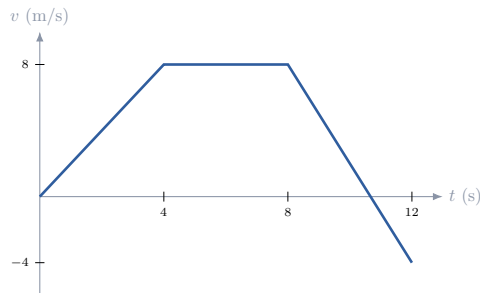
Independent problem

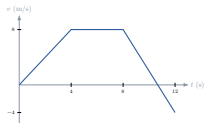
For the motion on the right, find:

- (a) the acceleration in each interval and sketch the a_x-t graph;
- (b) the displacement from 0 to 12 s;
- (c) the distance travelled.

Predict first

Is the displacement larger or smaller than the distance? When does the object reverse?





Equations: $a_x = \Delta v_x / \Delta t$ (slope); $\Delta x = \int v_x dt$ (signed area).

Plan: slopes for (a); areas for (b), split at $v_x = 0$ for (c).

(a) Slopes:

$$0-4 \text{ s: } a_x = +8/4 = +2.0 \text{ m/s}^2$$

$$4-8 \text{ s: } a_x = 0$$

$$8-12 \text{ s: } a_x = -12/4 = -3.0 \text{ m/s}^2$$

The a_x - t graph is three horizontal steps: +2, 0, -3.

(b), (c) Areas:

$$0-4: \frac{1}{2}(4)(8) = +16 \text{ m}$$

$$4-8: 4 \times 8 = +32 \text{ m}$$

$$8-10.67: +10.7 \text{ m (until } v_x = 0)$$

$$10.67-12: -2.7 \text{ m}$$

Result

Displacement = $16 + 32 + 10.7 - 2.7 = +56 \text{ m}$; distance = $16 + 32 + 10.7 + 2.7 = 61.3 \text{ m}$.

It reverses at $t = 10.67 \text{ s}$ ($v_x = 0$). Area below the axis is negative displacement but positive distance.

But what if... the graph is a smooth curve $a_x(t)$, given by a formula instead of straight segments? The area becomes an **integral**.

Rule: from acceleration back to position

$$v_x(t) = v_{0x} + \int_0^t a_x dt' \quad x(t) = x_0 + \int_0^t v_x dt'$$

Add the initial conditions v_{0x} and x_0 ; without them the integral is defined only up to a constant. These integrals are the areas under a_x-t (giving Δv_x) and v_x-t (giving Δx).

When to use it

The acceleration (or velocity) is given as a function of time and you need the velocity and position.

Watch out and check

Do not forget the initial value. Check $v(0) = v_0$ and $x(0) = x_0$; differentiate your answer to recover the $a(t)$ you started with.

Question

A particle on the x -axis starts at $x_0 = 2.0$ m with $v_{0x} = -3.0$ m/s. Its acceleration is $a_x = (6.0t - 4.0)$ m/s². Find $v_x(t)$, $x(t)$, and the values at $t = 3.0$ s.

Predict first

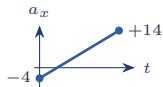
At $t = 0$, $v_x < 0$ and $a_x < 0$. Is it speeding up or slowing down at first? At $t = 3$ s, what is the sign of a_x ?

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Predict first

At $t = 0$, $v_x < 0$ and $a_x < 0$. Is it speeding up or slowing down at first? At $t = 3$ s, what is the sign of a_x ?



Equations: $v_x = v_{0x} + \int_0^t a_x dt'$, $x = x_0 + \int_0^t v_x dt'$. **Plan:** integrate twice, then put $t = 3.0$ s.

Solve: $v_x = -3.0 + \int_0^t (6.0t' - 4.0)dt' = -3.0 - 4.0t + 3.0t^2$

$x = 2.0 + \int_0^t (-3.0 - 4.0t' + 3.0t'^2)dt' = 2.0 - 3.0t - 2.0t^2 + t^3$

At $t = 3.0$ s: $v_x = -3 - 12 + 27 = +12$ m/s, $x = 2 - 9 - 18 + 27 = 2.0$ m, $a_x = +14$ m/s².

Evaluate: v_x and a_x have the same sign at $t = 0$ (speeding up toward $-x$); v_x then reverses and the particle returns to $x = 2.0$ m. Differentiating x twice gives $6.0t - 4.0$. ✓

Decide before any equation

1. Can the velocity be zero while the acceleration is not?
2. Can the acceleration be constant while the velocity changes sign?
3. If a_x is constant and non-zero, can v_x stay constant?

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1. Can the velocity be zero while the acceleration is not?
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3. If a_x is constant and non-zero, can v_x stay constant?

Answers

1. **Yes.** At the top of a thrown ball, $v_y = 0$ but $a_y = -g$.
2. **Yes.** Constant $a_x = -2 \text{ m/s}^2$ takes v_x from $+4$ through 0 to -4 m/s .
3. **No.** $a_x = dv_x/dt$; constant velocity means zero acceleration.

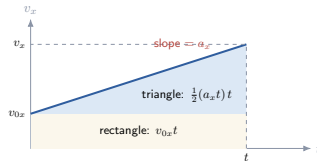
But what if... the acceleration is *constant*, as in these answers? Then the integrals are the same every time, and we can skip the calculus.

Rule: valid only for constant acceleration along a straight line

- | | | |
|-----|--|--------------|
| (1) | $v_x = v_{0x} + a_x t$ | no $x - x_0$ |
| (2) | $x - x_0 = v_{0x} t + \frac{1}{2} a_x t^2$ | no v_x |
| (3) | $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$ | no t |
| (4) | $x - x_0 = \frac{1}{2}(v_{0x} + v_x) t$ | no a_x |
| (5) | $x - x_0 = v_x t - \frac{1}{2} a_x t^2$ | no v_{0x} |

Five variables appear: $x - x_0$, v_{0x} , v_x , a_x and t . Each equation omits exactly one. **To choose:** confirm a_x is constant and the motion is on a line; list the knowns and the target; find the variable that is *neither known nor wanted*; use the equation that omits it; solve algebraically, then substitute with units.

But what if... we apply these to the previous example, where $a_x = 6.0t - 4.0$ is not constant? Test it with numbers.



If a_x is constant, v_x is a straight line on a v_x-t graph, and the displacement is the area under it: $v_{0x}t$ (rectangle) + $\frac{1}{2}a_x t^2$ (triangle). Eliminating t gives (3); the trapezoid gives (4). Derivations are in the supplementary notes.

Apply the constant-acceleration equations to the changing-acceleration example

There $a_x = 6.0t - 4.0$ varies. At $t = 3.0$ s, $a_x = 14$ m/s². Suppose we wrongly used that value as if it were constant:

Wrong (constant $a_x = 14$):

$$v_x = -3.0 + 14(3.0) = +39 \text{ m/s}$$

$$x = 2.0 + (-3.0)(3.0) + \frac{1}{2}(14)(9.0) = 56 \text{ m}$$

Correct (integration):

$$v_x = +12 \text{ m/s}$$

$$x = 2.0 \text{ m}$$

Lesson

Before using equations (1)–(5), ask: *is a_x constant?* If not, integrate.

But what if... a_x *is* constant, and there are five equations to choose from? Pick the one that omits the variable you neither know nor want.

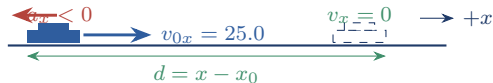
Question

A car moving at 25.0 m/s (90 km/h) brakes with a constant acceleration of magnitude 6.0 m/s^2 until it stops. How far does it travel, and how long does it take?

Predict first

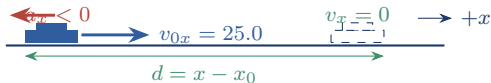
Sketch with $+x$ along the motion. Sign of a_x ? A rough distance if the average speed is about half the initial speed and the stop takes about 4 s ?





Equations: (3) $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$ and (1) $v_x = v_{0x} + a_x t$.

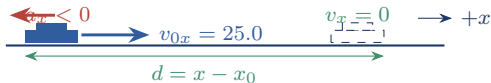
Plan: $+x$ along the motion; $v_{0x} = +25.0$ m/s, $v_x = 0$, $a_x = -6.0$ m/s². Equation (3) omits t , so use it for the distance; (1) omits $x - x_0$, so use it for the time.



Equations: (3) $v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$ and (1) $v_x = v_{0x} + a_xt$.

Plan: $+x$ along the motion; $v_{0x} = +25.0 \text{ m/s}$, $v_x = 0$, $a_x = -6.0 \text{ m/s}^2$. Equation (3) omits t , so use it for the distance; (1) omits $x - x_0$, so use it for the time.

Solve: $x - x_0 = \frac{v_x^2 - v_{0x}^2}{2a_x} = \frac{0 - 625}{2(-6.0)} = 52 \text{ m}$ $t = \frac{v_x - v_{0x}}{a_x} = \frac{0 - 25.0}{-6.0} = 4.2 \text{ s}$



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Evaluate: average speed $= \frac{1}{2}(25.0 + 0) = 12.5 \text{ m/s}$, and $12.5 \times 4.17 = 52 \text{ m}$, matching equation (4). Units m and s. The estimate ($\approx 50 \text{ m}$) agrees.

Physical meaning

$x - x_0 \propto v_{0x}^2$: doubling the speed quadruples the braking distance.

But what if... the acceleration is the same for every object near the ground, such as a ball thrown upward that rises, stops and falls? One value of a for the whole flight. $\rightarrow$ **free fall**.

Rule: free fall

Near the Earth's surface, neglecting air resistance, all objects have the same acceleration:

$$a_y = -g = -9.80 \text{ m/s}^2 \quad (+y \text{ upward})$$

It holds **throughout** the motion: rising, at the top, and falling.

Predict first

A hammer and a feather are dropped in vacuum. Which lands first?

Misconception

“At the top the acceleration is zero.” It is not: $v_y = 0$ momentarily, but $a_y = -g$ still.

Physical meaning

g is the *magnitude*; the sign of a_y comes from the axis you chose .
With $+y$ downward, $a_y = +g$.

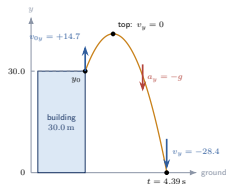
Question

A ball is thrown **upward** at 14.7 m/s from the edge of a 30.0 m building. How long until it hits the ground, and what is its velocity then? (Neglect air resistance.)

Predict first

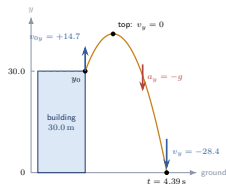
Sketch. Is the time more or less than the 3.00 s of a throw from the ground? Sign of v_y at impact?

But what if... a ball thrown upward at 14.7 m/s from the ground is back at its start after 3.00 s , with time up = time down? From the edge of a building it lands *below* its launch point, and the landing-time equation is a quadratic with two roots. Which one is physical?



Equations: (2) $y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$ and (1) $v_y = v_{0y} + a_yt$.

Plan: $+y$ up, origin at the ground: $y_0 = 30.0 \text{ m}$, $y = 0$, $v_{0y} = +14.7 \text{ m/s}$, $a_y = -9.80 \text{ m/s}^2$. Solve (2) for t (a quadratic), keep the physical root, then use (1) for v_y .

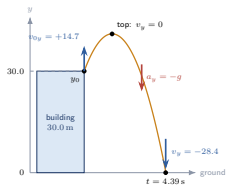


Equations: (2) $y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$ and (1) $v_y = v_{0y} + a_yt$.

Plan: $+y$ up, origin at the ground: $y_0 = 30.0$ m, $y = 0$, $v_{0y} = +14.7$ m/s, $a_y = -9.80$ m/s². Solve (2) for t (a quadratic), keep the physical root, then use (1) for v_y .

Solve: (2): $0 = 30.0 + 14.7t - 4.9t^2 \Rightarrow 4.9t^2 - 14.7t - 30.0 = 0$

$$t = \frac{14.7 \pm \sqrt{14.7^2 + 4(4.9)(30.0)}}{2(4.9)} = 4.39 \text{ s or } -1.39 \text{ s}$$



Equations: (2) $y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$ and (1) $v_y = v_{0y} + a_yt$.

Plan: $+y$ up, origin at the ground: $y_0 = 30.0$ m, $y = 0$, $v_{0y} = +14.7$ m/s, $a_y = -9.80$ m/s². Solve (2) for t (a quadratic), keep the physical root, then use (1) for v_y .

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$$t = \frac{14.7 \pm \sqrt{14.7^2 + 4(4.9)(30.0)}}{2(4.9)} = 4.39 \text{ s or } -1.39 \text{ s}$$

Only $t = 4.39$ s is physical ($t > 0$). Then
 $v_y = 14.7 - 9.8(4.39) = -28.4$ m/s.

Evaluate: longer than 3.00 s as predicted; $v_y < 0$ downward; check (3):
 $v_y^2 = 14.7^2 + 2(9.8)(30.0) = 804$, so $|v_y| = 28.4$ m/s ✓

Interpretation of the negative root

$t = -1.39$ s is the time at which the same parabola would have crossed $y = 0$ *before* the throw. Mathematically valid; physically rejected.

Common mistakes and the correct habit

Mistake

Thinking a negative v_x or a negative a_x means slowing down.

Using the constant-acceleration equations when a_x varies.

Setting $a_y = 0$ at the top of a flight.

Keeping the quadratic root with $t < 0$.

Integrating without the initial conditions.

Putting $-g$ into an axis that points down.

Instead

Compare the two signs: the same sign means speeding up.

Ask whether a_x is constant; if not, integrate.

$a_y = -g$ throughout; only v_y is zero there.

Choose the physical root and say why.

Add v_{0x} and x_0 , then check $v(0)$ and $x(0)$.

The sign of a_y comes from the axis you chose.

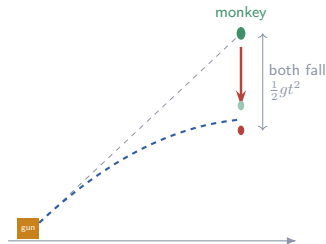
Contact 3: Apply

How do the same ideas describe more complex motion?

We have modelled motion along a line. **What changes when the motion is not along a line, or when the observer is moving?** Projectiles, a car on a bend, a walker on a train. **No new kinematics is needed:** this contact reuses the equations of Contacts 1 and 2, component by component. Start with a puzzle.

The monkey and the dart

A dart is aimed directly at a monkey hanging in a tree. When the dart leaves the gun, the monkey lets go and falls. Does the dart hit, pass above, or pass below the monkey? (Assume the dart reaches the monkey's line before the monkey reaches the ground.) Commit to an answer: which of the two experiences gravity?



But what if... the dart moves forward *and* falls at once? We need to know how the horizontal and vertical motions combine.

Rule: independence of components

With uniform gravity and air resistance neglected, the horizontal and vertical component equations can be solved independently. They are linked **only through the common time t** .

Why the dart hits:

Without gravity, both the dart and the monkey would stay on the line of sight. With gravity, both fall the *same* distance $\frac{1}{2}gt^2$ below it in time t . The dart still meets the monkey.

Misconception

No horizontal force acts on a projectile when air resistance is neglected. $a_x = 0$, so v_x is constant.

But what if... we want the height, the time of flight and the range, as numbers? Write both component motions as a **model**.

Assumptions: neglect air resistance; g constant; flat Earth over the range.

Horizontal ($a_x = 0$):

$$v_x = v_0 \cos \theta$$

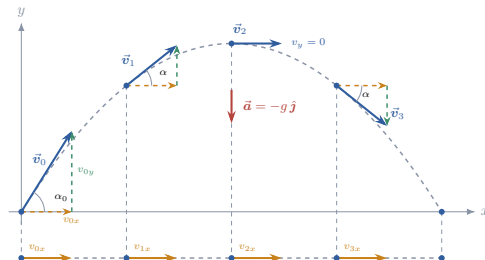
$$x = x_0 + v_0 \cos \theta t$$

Vertical ($a_y = -g$):

$$v_y = v_0 \sin \theta - gt$$

$$y = y_0 + v_0 \sin \theta t - \frac{1}{2}gt^2$$

At the top: $v_y = 0$ but $a_y = -g$.



Method

Sketch (start, top, end); axes $+x$ horizontal, $+y$ up; resolve $\vec{v}_0$; x -motion with $a_x = 0$; y -motion with $a_y = -g$ *Reuse*: (Contact 2 equations (1), (2), once per axis); **same time** in both; reconstruct speed and direction if needed. Quadratic in t : keep the physical root.

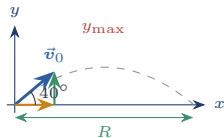
But what if... we try it on a real launch from level ground? Predict first, then calculate.

Question

A ball is hit from ground level at 30.0 m/s , 40.0° above the horizontal. Find the time to the highest point, the maximum height, the total time of flight and the horizontal range. (Neglect air resistance; the ground is level.)

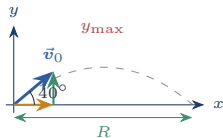
Predict first

Sketch. At 45° the range would be $v_0^2/g \approx 92 \text{ m}$; is the range at 40° slightly smaller or larger? What are v_y and a_y at the top?



Equations: horizontal $x = v_{0x}t$ ($a_x = 0$); vertical $y = v_{0y}t - \frac{1}{2}gt^2$, $v_y = v_{0y} - gt$. **Reuse:** Contact 2 (1), (2) applied to x and y .

Plan: resolve $\vec{v}_0$; top: $v_y = 0$; landing: $y = 0$; range: put the flight time into $x = v_{0x}t$.



Equations: horizontal $x = v_{0x}t$ ($a_x = 0$); vertical $y = v_{0y}t - \frac{1}{2}gt^2$, $v_y = v_{0y} - gt$. **Reuse:** Contact 2 (1), (2) applied to x and y .

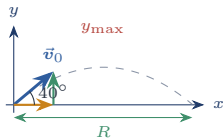
Plan: resolve $\vec{v}_0$; top: $v_y = 0$; landing: $y = 0$; range: put the flight time into $x = v_{0x}t$.

Horizontal: $v_{0x} = 30.0 \cos 40.0^\circ = 23.0 \text{ m/s}$, constant. Range $x = 23.0 \times 3.94 = 90.4 \text{ m}$ (T from below, unrounded values).

Vertical: $v_{0y} = 30.0 \sin 40.0^\circ = 19.3 \text{ m/s}$. Top: $v_y = 0 \Rightarrow t = \frac{19.3}{9.80} = 1.97 \text{ s}$;

$$y_{\max} = \frac{19.3^2}{2(9.80)} = 19.0 \text{ m}.$$

Landing: $y = 0 \Rightarrow T = \frac{2(19.3)}{9.80} = 3.94 \text{ s}$ (the other root is $t = 0$).



Equations: horizontal $x = v_{0x}t$ ($a_x = 0$); vertical $y = v_{0y}t - \frac{1}{2}gt^2$, $v_y = v_{0y} - gt$. **Reuse:** Contact 2 (1), (2) applied to x and y .

Plan: resolve $\vec{v}_0$; top: $v_y = 0$; landing: $y = 0$; range: put the flight time into $x = v_{0x}t$.

Horizontal: $v_{0x} = 30.0 \cos 40.0^\circ = 23.0 \text{ m/s}$, constant. Range $x = 23.0 \times 3.94 = 90.4 \text{ m}$ (T from below, unrounded values).

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Landing: $y = 0 \Rightarrow T = \frac{2(19.3)}{9.80} = 3.94 \text{ s}$ (the other root is $t = 0$).

Evaluate: range slightly less than 92 m, as predicted. At the top $\vec{v} = (23.0 \hat{i} + 0 \hat{j}) \text{ m/s}$ and $a_y = -9.80 \text{ m/s}^2$. Total time is twice the time to the top, as expected for a symmetric path. ✓Landing *below* the launch height breaks the symmetry: use the quadratic, as in Example 4.

But what if... gravity changes both the size and the direction of $\vec{v}$? What if only the *direction* changes, at constant speed? $\rightarrow$ **circular motion.**

Poll

A car goes around a circular track at a **constant speed**. Is its acceleration zero?

Predict first

Recall: acceleration is the rate of change of the *velocity vector*. Does the velocity vector stay the same?

Poll

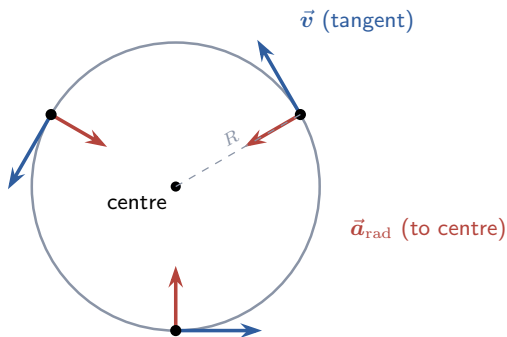
A car goes around a circular track at a **constant speed**. Is its acceleration zero?

Predict first

Recall: acceleration is the rate of change of the *velocity vector*. Does the velocity vector stay the same?

No. The speed is constant but the direction of $\vec{v}$ changes continuously, so $\vec{a} \neq 0$. This is $a_{\perp}$ from Contact 2, with no $a_{\parallel}$.

But what if... $\vec{a} \neq 0$, but in which direction does it point? Draw $\vec{v}$ at two nearby points.

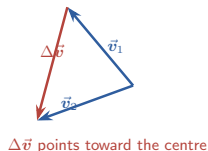
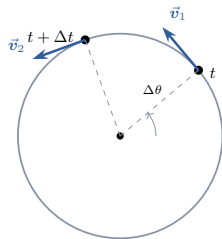


- $\vec{v}$ is tangent to the circle, with constant magnitude v .
- The acceleration has no tangential part: $a_{\text{tan}} = 0$.
- The acceleration points toward the centre: radial, or **centripetal**.

Predict first

If the speed doubles at the same radius, does the acceleration double or quadruple?

But what if... we want its size? Compare $\vec{v}$ at two nearby instants.



In a short time Δt the velocity turns through an angle $\Delta\theta$. The arc is $v \Delta t = R \Delta\theta$, and $|\Delta \vec{v}| \approx v \Delta\theta$ points toward the centre.

Reuse: $\vec{a} = \Delta \vec{v} / \Delta t$ from Contact 2. So

$$a_{\text{rad}} = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{v \Delta\theta}{\Delta t} = \frac{v \cdot v \Delta t / R}{\Delta t}$$

Rule: radial acceleration

$$a_{\text{rad}} = \frac{v^2}{R}$$

But what if... we are given the number of turns per second, not v ? Link v to the period and frequency.

Build the relations from one revolution:

angle 2π rad, distance $2\pi R$

- **Period** T : time for one revolution.
- **Frequency** $f = 1/T$ (hertz, $\text{Hz} = \text{s}^{-1}$).
- **Angular speed** $\omega = 2\pi f = 2\pi/T$ (rad/s).
- **Speed**: $v = \frac{2\pi R}{T} = \omega R$. *Reuse*: (distance over time, Contact 1).

Rule: uniform circular motion

$$a_{\text{rad}} = \frac{v^2}{R} = \omega^2 R = \frac{4\pi^2 R}{T^2}$$

Physical meaning

Constant speed does not mean zero acceleration. For motion on a circle at constant speed, $\vec{a}$ points to the centre and has magnitude v^2/R .

Question

A sample at radius $R = 0.10$ m in a centrifuge rotates at 3000 rev/min. Find the period, the angular speed, the speed of the sample and its radial acceleration in multiples of g .

Predict first

Order of magnitude: much smaller than, similar to, or much larger than g ?

Question

A sample at radius $R = 0.10$ m in a centrifuge rotates at 3000 rev/min. Find the period, the angular speed, the speed of the sample and its radial acceleration in multiples of g .

Predict first

Order of magnitude: much smaller than, similar to, or much larger than g ?



Equations: $T = 1/f$, $\omega = 2\pi f$, $v = \omega R$, $a_{\text{rad}} = \omega^2 R$ **Reuse:** (speed and acceleration definitions, Contacts 1 and 2). **Plan:** Hz first, then in order.

Solve: $f = 3000/60 = 50$ Hz, $T = 1/f = 0.020$ s, $\omega = 2\pi f = 314$ rad/s, $v = \omega R = 31.4$ m/s.
 $a_{\text{rad}} = \omega^2 R = (314)^2(0.10) = 9.9 \times 10^3$ m/s² $\approx 1.0 \times 10^3 g$.

Evaluate: $v^2/R = 31.4^2/0.10 = 9.9 \times 10^3$ ✓; far larger than g , as expected.

But what if... a passenger walks along the aisle of the car, or of a train, while it moves? Then we must ask *whose* velocity we mean. → **relative velocity**.

Discussion

A passenger walks along the aisle of a train moving at 20.0 m/s relative to the ground. How fast is the passenger moving?

Predict first

The passenger walks toward the front at 1.5 m/s relative to the train. Faster or slower than the train, relative to the ground? What if the passenger walks toward the rear?

Discussion

A passenger walks along the aisle of a train moving at 20.0 m/s relative to the ground. How fast is the passenger moving?

Predict first

The passenger walks toward the front at 1.5 m/s relative to the train. Faster or slower than the train, relative to the ground? What if the passenger walks toward the rear?

The same motion has different velocities for different observers.

But what if... the question has no single answer until we say *relative to what?* We need a notation that records whose velocity, and relative to which observer.

Rule: relative velocity

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Read $\vec{v}_{P/A}$ as “the velocity of P relative to A ”.
Also $\vec{v}_{A/B} = -\vec{v}_{B/A}$. The inner subscript B
cancels in the chain: P/B followed by B/A gives
 P/A . Write the relation *before* inserting values. In
2D, use components and reconstruct (Week 1).

Reuse: vector addition from Contact 1.

Watch out

Do not add speeds: add *vectors*.

Back to the train: P, T, G

$$v_{T/G,x} = +20.0 \text{ m/s; toward the front,}$$
$$v_{P/T,x} = +1.5 \text{ m/s.}$$

Rule: relative velocity

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Read $\vec{v}_{P/A}$ as “the velocity of P relative to A ”. Also $\vec{v}_{A/B} = -\vec{v}_{B/A}$. The inner subscript B cancels in the chain: P/B followed by B/A gives P/A . Write the relation *before* inserting values. In 2D, use components and reconstruct (Week 1).

Reuse: vector addition from Contact 1.

Watch out

Do not add speeds: add *vectors*.

But what if... the velocities are not along one line, such as a boat crossing a flowing river? Use components.

Back to the train: P, T, G

$v_{T/G,x} = +20.0 \text{ m/s}$; toward the front,
 $v_{P/T,x} = +1.5 \text{ m/s}$.

$$v_{P/G,x} = v_{P/T,x} + v_{T/G,x} = 1.5 + 20.0 = 21.5 \text{ m/s}.$$

Toward the rear: $v_{P/T,x} = -1.5 \text{ m/s}$, so
 $v_{P/G,x} = 18.5 \text{ m/s}$: still forward relative to the ground, slower than the train.

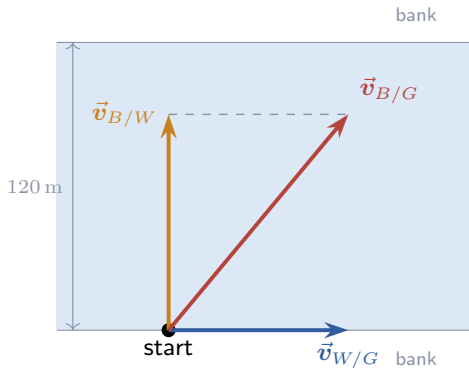
Evaluate: the signs follow from the chosen $+x$.

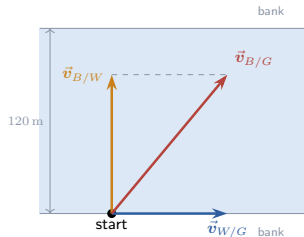
Question

A river flows east at 3.0 m/s relative to the ground and is 120 m wide. A boat has speed 4.0 m/s relative to the water and is pointed due north. Find its velocity relative to the ground, the crossing time and where it lands.

Predict first

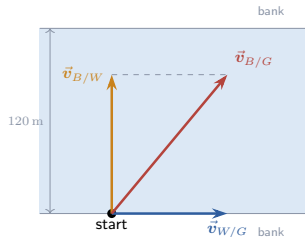
Which way is the boat swept? Is the ground speed more or less than 4.0 m/s ?





Equations: $\vec{v}_{B/G} = \vec{v}_{B/W} + \vec{v}_{W/G}$; crossing $t = d/v_y$; drift $= v_x t$. **Reuse:** $x = v_x t$ from Contact 2 with $a = 0$, for each component.

Plan: x east, y north; $\vec{v}_{W/G} = 3.0 \hat{i}$, $\vec{v}_{B/W} = 4.0 \hat{j}$ (m/s). Add the vectors, then use the northward component for the time.

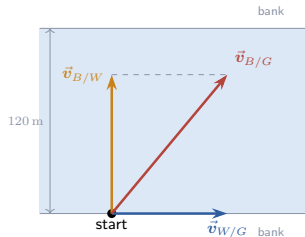


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Solve: $\vec{v}_{B/G} = (3.0 \hat{i} + 4.0 \hat{j})$ m/s, so $v_{B/G} = 5.0$ m/s at $\tan^{-1}(3.0/4.0) = 36.9^\circ$ east of north.

Crossing time depends only on the northward motion: $t = \frac{120}{4.0} = 30$ s. Downstream drift:
 $3.0 \times 30 = 90$ m.



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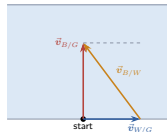
Crossing time depends only on the northward motion: $t = \frac{120}{4.0} = 30$ s. Downstream drift:
 $3.0 \times 30 = 90$ m.

Evaluate: swept east, ground speed $5.0 > 4.0$ m/s, as predicted; the current adds sideways motion but does not change the crossing time. $\sqrt{90^2 + 120^2} = 150$ m $= 5.0 \times 30$ ✓

But what if... the boat must land at the point *directly opposite*? Pointing north would be swept away. Which way must we point?

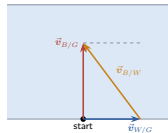
Independent problem

The same river (3.0 m/s east, 120 m wide). A boat with speed 5.0 m/s relative to the water must reach the point directly opposite its start. Which way must it be pointed, and how long does the crossing take? Predict: upstream or downstream?



Independent problem

The same river (3.0 m/s east, 120 m wide). A boat with speed 5.0 m/s relative to the water must reach the point directly opposite its start. Which way must it be pointed, and how long does the crossing take? Predict: upstream or downstream?



Equation: $\vec{v}_{B/G} = \vec{v}_{B/W} + \vec{v}_{W/G}$, with $\vec{v}_{B/G}$ due north, so $v_{B/G,x} = 0$.

Plan: write $\vec{v}_{B/W}$ at angle φ west of north; the x equation gives φ , the y equation gives the crossing speed.

Solve: x : $-5.0 \sin \varphi + 3.0 = 0 \Rightarrow \varphi = 36.9^\circ$ **west of north** (upstream). y :

$v_{B/G,y} = 5.0 \cos 36.9^\circ = 4.0$ m/s, so $t = 120/4.0 = 30$ s.

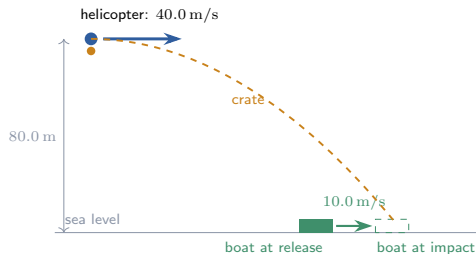
Evaluate: upstream, as predicted; ground speed $4.0 < 5.0$ m/s because part of the boat's speed cancels the current; the same triangle as Example 7. ✓

But what if... a problem uses several of these ideas at once, such as a falling object seen from a moving boat? Which model, and when? → **integrated applications.**

Application problem: decide what physics applies

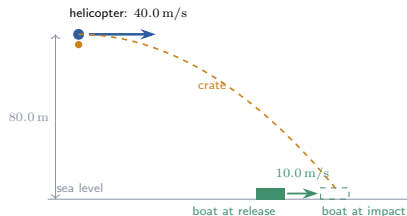
A helicopter flies horizontally at 40.0 m/s , 80.0 m above the sea. A boat below moves in the same direction at 10.0 m/s . A crate released from the helicopter is to land on the boat.

- How far ahead of the crate (horizontally) must the boat be at release?
- With what speed does the crate hit the boat, as seen from the boat?



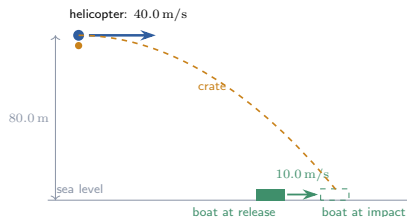
Predict first

IDENTIFY → SET UP (frame? axes? sketch) → PREDICT → EXECUTE → EVALUATE.



Identify: the crate is a projectile; the boat is another frame. **Equations:** horizontal $x_{C/B} = v_{C/B,x} t$ with $\vec{v}_{C/B} = \vec{v}_{C/G} - \vec{v}_{B/G}$; vertical $y = y_0 - \frac{1}{2}gt^2$, $v_y = -gt$. **Reuse:** free fall from Contact 2, plus the relative-velocity rule above.

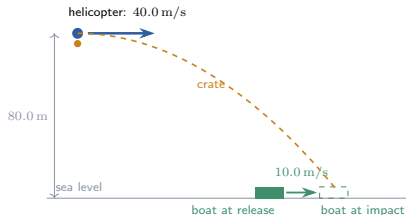
Plan: $+x$ forward, $+y$ up; $\vec{v}_{C/G} = (40.0, 0)$ at release. Find t from the vertical fall, then the horizontal gain on the boat, then the relative velocity at impact.



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Horizontal: $v_{C/B,x} = 40.0 - 10.0 = 30.0 \text{ m/s}$ (relative-velocity rule), so with t from the vertical the boat must be $30.0 \times 4.04 = 121 \text{ m}$ ahead (a).



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Horizontal: $v_{C/B,x} = 40.0 - 10.0 = 30.0$ m/s (relative-velocity rule), so with t from the vertical the boat must be $30.0 \times 4.04 = 121$ m ahead (a).

Vertical: fall time from $y = 80.0$ m: $t = \sqrt{2(80.0)/9.80} = 4.04$ s;
 $v_{C/B,y} = -9.80(4.04) = -39.6$ m/s, so $v_{C/B} = \sqrt{30.0^2 + 39.6^2} = 49.7$ m/s (b).

Evaluate: units m and m/s; the crate relative to the boat has the same fall time but a smaller horizontal speed; 4.04 s is plausible for an 80 m fall ($\sqrt{2h/g}$). ✓

Common mistakes and the correct habit

Mistake**Instead**

A horizontal force on a projectile.

None is needed: $a_x = 0$ and v_x is constant (no air resistance).

Using different times for x and y .

The common time is the only link between the components.

Setting $a = 0$ at constant speed on a circle.

$a = v^2/R$ toward the centre.

Adding speeds in relative motion.

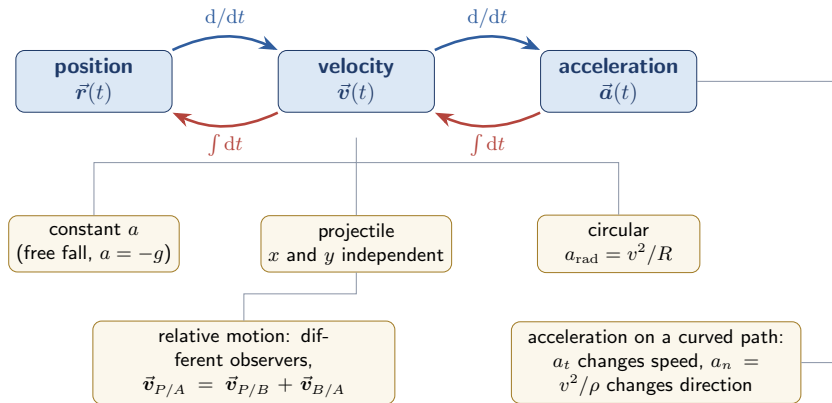
Add velocities as vectors with the subscript chain.

Using rev/min or degrees in $v = \omega R$.

Convert to rad/s first.

Setting $v_y = 0$ at landing.

$v_y = 0$ only at the top of the flight.



Key ideas

- **Position** is a vector from the origin you choose; displacement $\Delta \vec{r} = \vec{r}_2 - \vec{r}_1$ does not depend on the origin; distance travelled is a non-negative scalar.
- **Velocity** $\vec{v} = d\vec{r}/dt$ is a vector tangent to the path; speed $v = |\vec{v}|$ is its magnitude; v_x is a signed number.
- **Acceleration** $\vec{a} = d\vec{v}/dt$: the part parallel to $\vec{v}$ changes the speed; the perpendicular part changes the direction.
- Slope and area link the $x-t$, $v-t$, $a-t$ graphs. Along a line, v_x and a_x with the same sign mean speeding up.
- **The constant-acceleration equations hold only when a_x is constant along a line; otherwise integrate, with initial conditions.**
- Free fall without air resistance: $a_y = -g$ throughout the flight, including the top.
- **In projectile motion (no air resistance) the horizontal and vertical motions are independent and linked only through the time.**
- Constant speed on a circle still means $a = v^2/R \neq 0$, directed to the centre.
- Velocities add as vectors, with the subscript chain $\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$.
- Habit: sketch, predict, calculate, evaluate.

Reference tools: the next slides (master toolkit, core equations). The Reference deck holds the toolkits and practice.

✂ MASTER TOOLKIT – For a motion problem

1. **IDENTIFY** Which object, which interval, what is asked?
 2. **FRAME** Relative to which observer?
 3. **CHOOSE AXES** Origin and positive directions.
 4. **SKETCH** Trajectory or motion diagram; add $\vec{v}$, $\vec{a}$.
 5. **PREDICT** Signs, direction, trend, rough size.
 6. **SELECT MODEL** General $\vec{r}$, $\vec{v}$, $\vec{a}$? Constant a ? Free fall? Projectile? Circular? Relative?
 7. **EXECUTE** Key move for the model (right); signs and units.
 8. **EVALUATE** Units, sign, direction, graph, assumptions, reasonableness.
- If the result disagrees with the prediction, investigate. Seven toolkits: Reference deck.

Step 7: the key move for each model

Formulas in t : differentiate each component; integrate with initial conditions.

Graph: slope gives the rate, area the change.

Constant a : use the equation that omits the variable you neither know nor want.

Free fall: $a_y = -g$ throughout; keep the physical root.

Projectile: resolve; $a_x = 0$, $a_y = -g$; same time.

Circular: $a = v^2/R = \omega^2 R$ to the centre; SI units first.

Relative: $\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$ first; add vectors.

Framework: IDENTIFY (1–2), SET UP (3–6), EXECUTE (7), EVALUATE (8).

Idea	Equation	Conditions and units
Displacement	$\Delta x = x_2 - x_1, \quad \Delta \vec{r} = \vec{r}_2 - \vec{r}_1$	m; final minus initial
Velocity	$\vec{v} = d\vec{r}/dt, \quad v_{av,x} = \Delta x / \Delta t$	m/s; slope of $x-t$
Acceleration	$\vec{a} = d\vec{v}/dt = d^2\vec{r}/dt^2$	m/s ² ; slope of v_x-t
Speeding up	v_x and a_x same sign: speeds up; opposite: slows	motion along a line
Integration	$v_x = v_{0x} + \int a_x dt, \quad x = x_0 + \int v_x dt$	needs initial conditions
Constant a_x	$v_x = v_{0x} + a_x t, \quad v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	only if a_x is constant, straight line
Free fall	$a_y = -g, \quad g = 9.80 \text{ m/s}^2$	+y upward; no air resistance
Projectile	$a_x = 0, \quad a_y = -g; \text{ same } t$	$v_{0x} = v_0 \cos \theta, v_{0y} = v_0 \sin \theta$
Circular motion	$a_{rad} = v^2/R = \omega^2 R, \quad v = \omega R$	toward the centre; $\omega = 2\pi f = 2\pi/T$
Curved path	$a_t = dv/dt, \quad a_n = v^2/\rho$	speed and direction changes
Relative velocity	$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$	vector sum; m/s

Graphs, slope and area

The slope and area of motion graphs return in Lecture 4, where the area under a force–position graph is the work done. Lecture 3 relates the acceleration to the force: $a_x = F_x/m$.

Constant acceleration and free fall

$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$ becomes the work–energy theorem in Lecture 4; free fall leads to gravitational potential energy in Lecture 5.

Projectiles

Lecture 5 finds the speed at any height by energy conservation, with no time; Lecture 7 treats collisions in two dimensions with the same component method.

Circular motion

Lecture 3 supplies the force behind $a_{\text{rad}} = v^2/R$ (centripetal force); Lecture 8 extends ω , T and f to rotating bodies.

Relative velocity

Lecture 7: a collision looks different in different frames, and the velocity of the centre of mass is the natural frame in which to describe it.

Physical meaning

Independent components, the constant-acceleration equations, integration with initial conditions and $\vec{v} = d\vec{r}/dt$ remain the core of the motion toolbox for the rest of the course.

- ☐ tell displacement from distance, and velocity from speed, in a given journey?
- ☐ choose an origin and positive direction, and give each sign its meaning?
- ☐ find $\vec{v}$ and $\vec{a}$ from $\vec{r}(t)$ by differentiating each component?
- ☐ find $v_x(t)$ and $x(t)$ from $a_x(t)$ by integrating, with initial conditions?
- ☐ read slopes and areas on $x-t$, $v-t$ and $a-t$ graphs, and sketch one from a description?
- ☐ decide from the signs of v_x and a_x whether an object speeds up or slows down?
- ☐ choose a constant-acceleration equation, and show that a_x is constant first?
- ☐ solve a free-fall problem with a consistent sign convention, and reject the non-physical root?
- ☐ solve a projectile problem with x and y linked only by time?
- ☐ explain why uniform circular motion has acceleration, and calculate a_{rad} ?
- ☐ find a velocity relative to a second observer with the subscript chain?
- ☐ use IDENTIFY $\rightarrow$ SET UP $\rightarrow$ EXECUTE $\rightarrow$ EVALUATE, and choose the model in an unfamiliar problem?

Next

Tutorial sheet: the train and police-car problems and extra graph, free-fall, projectile, circular and relative-velocity problems. The Reference deck holds the supporting material cut from the live lecture; the supplementary notes hold derivations, the calculus refresher and the parabola extension.